CHAPTER 9

Sequences of Functions

1. Pointwise Convergence

We have accumulated much experience working with sequences of numbers. The next level of complexity is sequences of functions. This chapter explores several ways that sequences of functions can converge to another function. The basic starting point is contained in the following definitions.

DEFINITION 9.1. Suppose $S \subset \mathbb{R}$ and for each $n \in \mathbb{N}$ there is a function $f_n : S \to \mathbb{R}$. The collection $\{f_n : n \in \mathbb{N}\}$ is a *sequence* of functions defined on S.

For each fixed $x \in S$, $f_n(x)$ is a sequence of numbers, and it makes sense to ask whether this sequence converges. If $f_n(x)$ converges for each $x \in S$, a new function $f : S \to \mathbb{R}$ is defined by

$$f(x) = \lim_{n\to\infty} f_n(x).$$

The function f is called the *pointwise limit* of the sequence f_n, or, equivalently, it is said f_n *converges pointwise* to f. This is abbreviated $f_n \xrightarrow{S} f$, or simply $f_n \to f$, if the domain is clear from the context.

EXAMPLE 9.1. Let

$$f_n(x) = \begin{cases} 0, & x < 0 \\ x^n, & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}.$$

Then $f_n \to f$ where

$$f(x) = \begin{cases} 0, & x < 1 \\ 1, & x \ge 1 \end{cases}.$$

(See Figure 1.) This example shows that a pointwise limit of continuous functions need not be continuous.

EXAMPLE 9.2. For each $n \in \mathbb{N}$, define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \frac{nx}{1 + n^2x^2}.$$

(See Figure 2.) Clearly, each f_n is an odd function and $\lim_{|x|\to\infty} f_n(x) = 0$. A bit of calculus shows that $f_n(1/n) = 1/2$ and $f_n(-1/n) = -1/2$ are the extreme values of f_n. Finally, if $x \neq 0$,

$$|f_n(x)| = \left|\frac{nx}{1 + n^2x^2}\right| < \left|\frac{nx}{n^2x^2}\right| = \left|\frac{1}{nx}\right|$$

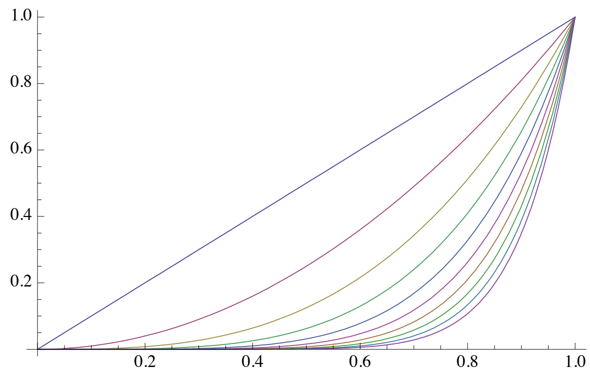

FIGURE 1. The first ten functions from the sequence of Example 9.1.

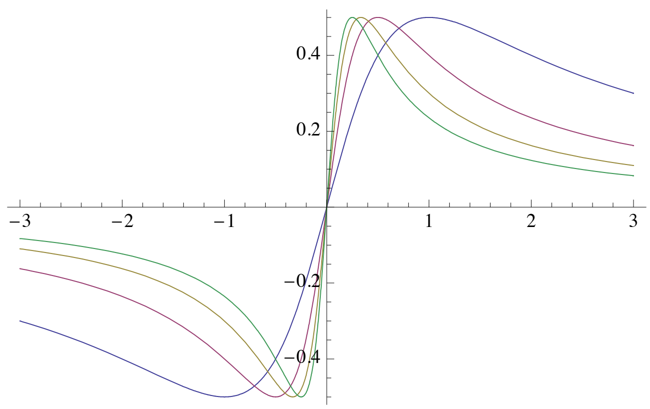

FIGURE 2. The first four functions from the sequence of Example 9.2.

implies $f_n \to 0$. This example shows that functions can remain bounded away from 0 and still converge pointwise to 0.

EXAMPLE 9.3. Define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 2^{2n+4}x - 2^{n+3}, & \frac{1}{2^{n+1}} < x < \frac{3}{2^{n+2}} \\ -2^{2n+4}x + 2^{n+4}, & \frac{3}{2^{n+2}} \le x < \frac{1}{2^n} \\ 0, & \text{otherwise} \end{cases}$$

To figure out what this looks like, it might help to look at Figure 3.

The graph of f_n is a piecewise linear function supported on $[1/2^{n+1}, 1/2^n]$ and the area under the isosceles triangle of the graph over this interval is 1. Therefore, $\int_0^1 f_n = 1$ for all n.

If $x > 0$, then whenever $x > 1/2^n$, we have $f_n(x) = 0$. From this it follows that $f_n \to 0$.

The lesson to be learned from this example is that it may not be true that $\lim_{n\to\infty} \int_0^1 f_n = \int_0^1 \lim_{n\to\infty} f_n$.

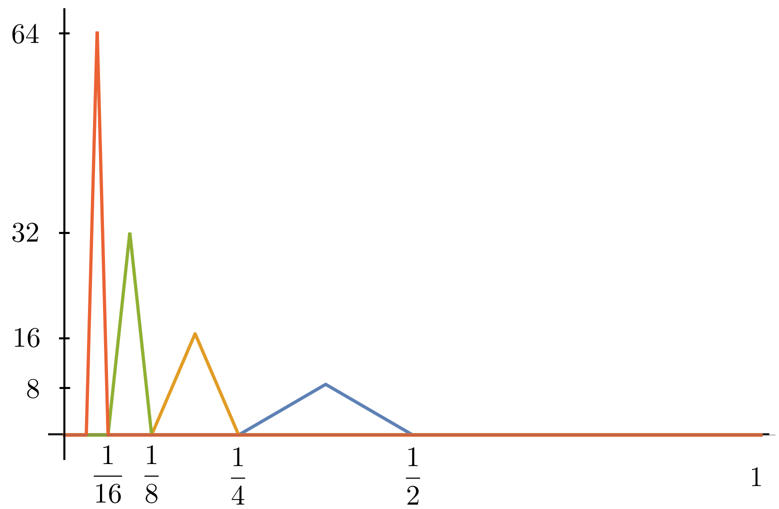

FIGURE 3. The first four functions $f_n \to 0$ from the sequence of Example 9.3.

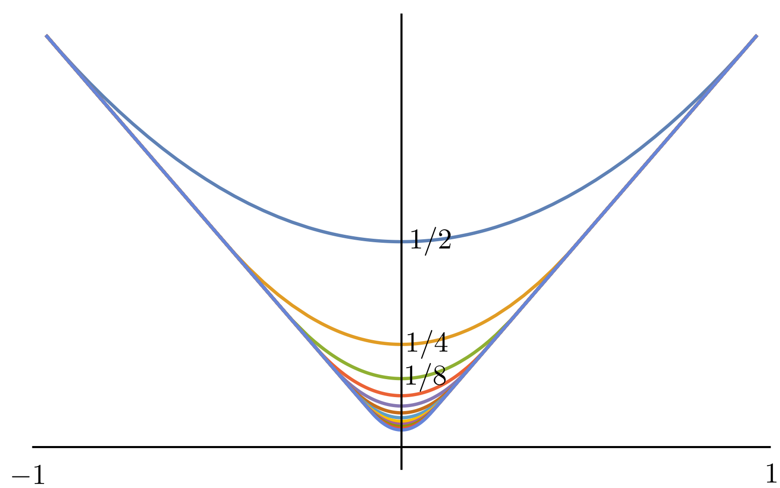

FIGURE 4. The first ten functions of the sequence $f_n \to |x|$ from Example 9.4.

EXAMPLE 9.4. Define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} \frac{n}{2}x^2 + \frac{1}{2n}, & |x| \le \frac{1}{n} \\ |x|, & |x| > \frac{1}{n} \end{cases}.$$

(See Figure 4.) The parabolic section in the center was chosen so $f_n(\pm 1/n) = 1/n$ and $f_n'(\pm 1/n) = \pm 1$. This splices the sections together at $(\pm 1/n, \pm 1/n)$ so f_n is differentiable everywhere. It's clear $f_n \to |x|$, which is not differentiable at 0.

This example shows that the limit of differentiable functions need not be differentiable.

The examples given above show that continuity, integrability and differentiability are not preserved in the pointwise limit of a sequence of functions. To have any hope of preserving these properties, a stronger form of convergence is needed.

2. Uniform Convergence

DEFINITION 9.2. The sequence $f_n : S \to \mathbb{R}$ *converges uniformly* to $f : S \to \mathbb{R}$ on S, if for each $\varepsilon > 0$ there is an $N \in \mathbb{N}$ so that whenever $n \geq N$ and $x \in S$, then $|f_n(x) - f(x)| < \varepsilon$.

In this case, we write $f_n \overset{S}{\rightrightarrows} f$, or simply $f_n \rightrightarrows f$, if the set S is clear from the context.

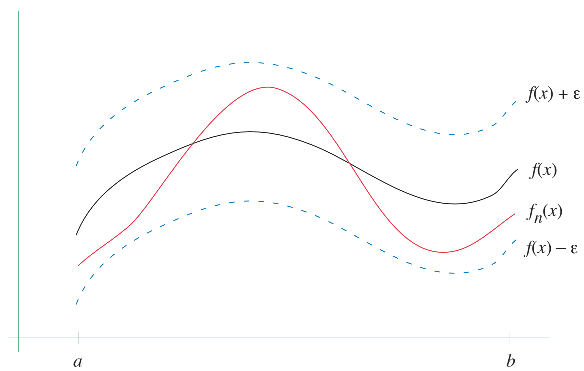

FIGURE 5. $|f_n(x) - f(x)| < \varepsilon$ on $[a, b]$, as in Definition 9.2.

The difference between pointwise and uniform convergence is that with pointwise convergence, the convergence of f_n to f can vary in speed at each point of S. With uniform convergence, the speed of convergence is roughly the same all across S. Uniform convergence is a stronger condition to place on the sequence f_n than pointwise convergence in the sense of the following theorem.

THEOREM 9.3. *If $f_n \overset{S}{\rightrightarrows} f$, then $f_n \xrightarrow{S} f$.*

PROOF. Let $x_0 \in S$ and $\varepsilon > 0$. There is an $N \in \mathbb{N}$ such that when $n \geq N$, then $|f(x) - f_n(x)| < \varepsilon$ for all $x \in S$. In particular, $|f(x_0) - f_n(x_0)| < \varepsilon$ when $n \geq N$. This shows $f_n(x_0) \to f(x_0)$. Since $x_0 \in S$ is arbitrary, it follows that $f_n \to f$. □

The first three examples given above show the converse to Theorem 9.3 is false. There is, however, one interesting and useful case in which a partial converse is true.

DEFINITION 9.4. If $f_n \xrightarrow{S} f$ and $f_n(x) \uparrow f(x)$ for all $x \in S$, then f_n *increases* to f on S. If $f_n \xrightarrow{S} f$ and $f_n(x) \downarrow f(x)$ for all $x \in S$, then f_n *decreases* to f on S. In either case, f_n is said to converge to f *monotonically*.

The functions of Example 9.4 decrease to $|x|$. Notice that in this case, the convergence is also happens to be uniform. The following theorem shows Example 9.4 to be an instance of a more general phenomenon.

THEOREM 9.5 (Dini's Theorem). *If*

(a) *S is compact,*
(b) *$f_n \xrightarrow{S} f$ monotonically,*
(c) *$f_n \in C(S)$ for all $n \in \mathbb{N}$, and*
(d) *$f \in C(S)$,*

then $f_n \rightrightarrows f$.

PROOF. There is no loss of generality in assuming $f_n \downarrow f$, for otherwise we consider $-f_n$ and $-f$. With this assumption, if $g_n = f_n - f$, then g_n is a sequence of continuous functions decreasing to 0. It suffices to show $g_n \rightrightarrows 0$.

To do so, let $\varepsilon > 0$. Using continuity and pointwise convergence, for each $x \in S$ find an open set G_x containing x and an $N_x \in \mathbb{N}$ such that $g_{N_x}(y) < \varepsilon$ for all $y \in G_x$. Notice that the monotonicity condition guarantees $g_n(y) < \varepsilon$ for every $y \in G_x$ and $n \geq N_x$.

The collection $\{G_x : x \in S\}$ is an open cover for S, so it must contain a finite subcover $\{G_{x_i} : 1 \leq i \leq n\}$. Let $N = \max\{N_{x_i} : 1 \leq i \leq n\}$ and choose $m \geq N$. If $x \in S$, then $x \in G_{x_i}$ for some i, and $0 \leq g_m(x) \leq g_N(x) \leq g_{N_i}(x) < \varepsilon$. It follows that $g_n \rightrightarrows 0$. □

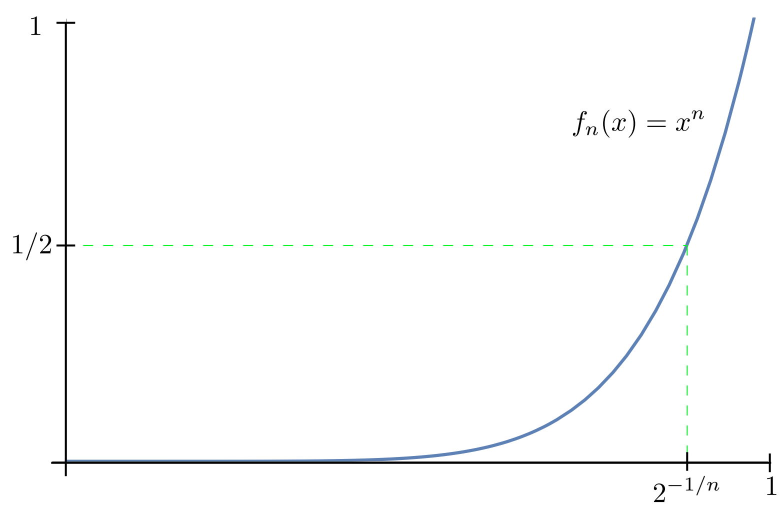

FIGURE 6. This shows a typical function from the sequence of Example 9.5.

EXAMPLE 9.5. Let $f_n(x) = x^n$ for $n \in \mathbb{N}$, then f_n decreases to 0 on $[0,1)$. If $0 < a < 1$ Dini's Theorem shows $f_n \rightrightarrows 0$ on the compact interval $[0,a]$. On the whole interval $[0,1)$, $f_n(x) > 1/2$ when $2^{-1/n} < x < 1$, so f_n is not uniformly convergent. (Why doesn't this violate Dini's Theorem?)

3. Metric Properties of Uniform Convergence

If $S \subset \mathbb{R}$, let $B(S) = \{f : S \to \mathbb{R} : f \text{ is bounded}\}$. For $f \in B(S)$, define $\|f\|_S = \operatorname{lub}\{|f(x)| : x \in S\}$. (It is abbreviated to $\|f\|$, if the domain S is clear from the context.) Apparently, $\|f\| \ge 0$, $\|f\| = 0 \iff f \equiv 0$ and, if $g \in B(S)$, then $\|f - g\| = \|g - f\|$. Moreover, if $h \in B(S)$, then

$$\begin{aligned} \|f-g\| &= \operatorname{lub}\{|f(x)-g(x)| : x \in S\} \\ &\le \operatorname{lub}\{|f(x)-h(x)| + |h(x)-g(x)| : x \in S\} \\ &\le \operatorname{lub}\{|f(x)-h(x)| : x \in S\} + \operatorname{lub}\{|h(x)-g(x)| : x \in S\} \\ &= \|f-h\| + \|h-g\| \end{aligned}$$

Combining all this, it follows that $\|f - g\|$ is a metric[1] on $B(S)$.

The definition of uniform convergence implies that for a sequence of bounded functions $f_n : S \to \mathbb{R}$,

$$f_n \rightrightarrows f \iff \|f_n - f\| \to 0.$$

Because of this, the metric $\|f - g\|$ is often called the *uniform metric* or the *sup-metric*. Many ideas developed using the metric properties of $\mathbb{R}$ can be carried over into this setting. In particular, there is a Cauchy criterion for uniform convergence.

Definition 9.6. Let $S \subset \mathbb{R}$. A sequence of functions $f_n : S \to \mathbb{R}$ is a *Cauchy sequence* under the uniform metric, if given $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that when $m, n \ge N$, then $\|f_n - f_m\| < \varepsilon$.

Theorem 9.7. *Let $f_n \in B(S)$. There is a function $f \in B(S)$ such that $f_n \rightrightarrows f$ iff f_n is a Cauchy sequence in $B(S)$.*

Proof. ($\Rightarrow$) Let $f_n \rightrightarrows f$ and $\varepsilon > 0$. There is an $N \in \mathbb{N}$ such that $n \ge N$ implies $\|f_n - f\| < \varepsilon/2$. If $m \ge N$ and $n \ge N$, then

$$\|f_m - f_n\| \le \|f_m - f\| + \|f - f_n\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

shows f_n is a Cauchy sequence.

($\Leftarrow$) Suppose f_n is a Cauchy sequence in $B(S)$ and $\varepsilon > 0$. Choose $N \in \mathbb{N}$ so that when $\|f_m - f_n\| < \varepsilon$ whenever $m \ge N$ and $n \ge N$. In particular, for a fixed $x_0 \in S$ and $m, n \ge N$, $|f_m(x_0) - f_n(x_0)| \le \|f_m - f_n\| < \varepsilon$ shows the sequence $f_n(x_0)$ is a Cauchy sequence in $\mathbb{R}$ and therefore converges. Since x_0 is an arbitrary point of S, this defines an $f : S \to \mathbb{R}$ such that $f_n \to f$.

Finally, if $m, n \ge N$ and $x \in S$ the fact that $|f_n(x) - f_m(x)| < \varepsilon$ gives

$$|f_n(x) - f(x)| = \lim_{m\to\infty} |f_n(x) - f_m(x)| \le \varepsilon.$$

This shows that when $n \ge N$, then $\|f_n - f\| \le \varepsilon$. We conclude that $f \in B(S)$ and $f_n \rightrightarrows f$. $\square$

[1] Definition 2.11

A collection of functions $\mathscr{S}$ is said to be *complete* under uniform convergence, if every Cauchy sequence in $\mathscr{S}$ converges to a function in $\mathscr{S}$. Theorem 9.7 shows $B(S)$ is complete under uniform convergence. We'll see several other collections of functions that are complete under uniform convergence.

EXAMPLE 9.6. For $S \subset \mathbb{R}$ let $L(S)$ be all the functions $f : S \to \mathbb{R}$ such that $f(x) = mx + b$ for some constants m and b. In particular, let f_n be a Cauchy sequence in $L([0,1])$. Theorem 9.7 shows there is an $f : [0,1] \to \mathbb{R}$ such that $f_n \rightrightarrows f$. In order to show $L([0,1])$ is complete, it suffices to show $f \in L([0,1])$.

To do this, let $f_n(x) = m_n x + b_n$ for each n. Then $f_n(0) = b_n \to f(0)$ and

$$m_n = f_n(1) - b_n \to f(1) - f(0).$$

Given any $x \in [0,1]$,

$$\begin{aligned} f_n(x) - ((f(1) - f(0))x + f(0)) &= m_n x + b_n - ((f(1) - f(0))x + f(0)) \\ &= (m_n - (f(1) - f(0)))x + b_n - f(0) \to 0. \end{aligned}$$

This shows $f(x) = (f(1) - f(0))x + f(0) \in L([0,1])$ and therefore $L([0,1])$ is complete.

EXAMPLE 9.7. Let $\mathscr{P} = \{p(x) : p \text{ is a polynomial}\}$. The sequence of polynomials $p_n(x) = \sum_{k=0}^{n} x^k/k!$ increases to e^x on $[0,a]$ for any $a > 0$, so Dini's Theorem shows $p_n \rightrightarrows e^x$ on $[0,a]$. But, $e^x \notin \mathscr{P}$, so $\mathscr{P}$ is not complete. (See Exercise 7.7.25.)

4. Series of Functions

The definitions of pointwise and uniform convergence are extended in the natural way to series of functions. If $\sum_{k=1}^{\infty} f_k$ is a series of functions defined on a set S, then the series converges pointwise or uniformly, depending on whether the sequence of partial sums, $s_n = \sum_{k=1}^{n} f_k$ converges pointwise or uniformly, respectively. It is absolutely convergent or absolutely uniformly convergent, if $\sum_{n=1}^{\infty} |f_n|$ is convergent or uniformly convergent on S, respectively.

The following theorem is obvious and its proof is left to the reader.

THEOREM 9.8. *Let $\sum_{n=1}^{\infty} f_n$ be a series of functions defined on S. If $\sum_{n=1}^{\infty} f_n$ is absolutely convergent, then it is convergent. If $\sum_{n=1}^{\infty} f_n$ is absolutely uniformly convergent, then it is uniformly convergent.*

The following theorem is a restatement of Theorem 9.5 for series.

THEOREM 9.9. *If $\sum_{n=1}^{\infty} f_n$ is a series of nonnegative continuous functions converging pointwise to a continuous function on a compact set S, then $\sum_{n=1}^{\infty} f_n$ converges uniformly on S.*

A simple, but powerful technique for showing uniform convergence of series is the following.

THEOREM 9.10 (Weierstrass M-Test). *If $f_n : S \to \mathbb{R}$ is a sequence of functions and M_n is a sequence nonnegative numbers such that $\|f_n\|_S \le M_n$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ is absolutely uniformly convergent.*

PROOF. Let $\varepsilon > 0$ and s_n be the sequence of partial sums of $\sum_{n=1}^{\infty} |f_n|$. Using the Cauchy criterion for convergence of a series, choose an $N \in \mathbb{N}$ such that when $n > m \geq N$, then $\sum_{k=m}^{n} M_k < \varepsilon$. So,

$$\|s_n - s_m\| = \|\sum_{k=m+1}^{n} f_k\| \leq \sum_{k=m+1}^{n} \|f_k\| \leq \sum_{k=m}^{n} M_k < \varepsilon.$$

This shows s_n is a Cauchy sequence and must converge according to Theorem 9.7. $\square$

EXAMPLE 9.8. Let $a > 0$ and $M_n = a^n / n!$. Since

$$\lim_{n\to\infty} \frac{M_{n+1}}{M_n} = \lim_{n\to\infty} \frac{a}{n+1} = 0,$$

the Ratio Test shows $\sum_{n=0}^{\infty} M_n$ converges. When $x \in [-a, a]$,

$$\left|\frac{x^n}{n!}\right| \leq \frac{a^n}{n!}.$$

The Weierstrass M-Test now implies $\sum_{n=0}^{\infty} x^n / n!$ converges absolutely uniformly on $[-a, a]$ for any $a > 0$. (See Exercise 9.4.)

5. Continuity and Uniform Convergence

THEOREM 9.11. *If $f_n : S \to \mathbb{R}$ is such that each f_n is continuous at x_0 and $f_n \overset{S}{\rightrightarrows} f$, then f is continuous at x_0.*

PROOF. Let $\varepsilon > 0$. Since $f_n \rightrightarrows f$, there is an $N \in \mathbb{N}$ such that whenever $n \geq N$ and $x \in S$, then $|f_n(x) - f(x)| < \varepsilon/3$. Because f_N is continuous at x_0, there is a $\delta > 0$ such that $x \in (x_0 - \delta, x_0 + \delta) \cap S$ implies $|f_N(x) - f_N(x_0)| < \varepsilon/3$. Using these two estimates, it follows that when $x \in (x_0 - \delta, x_0 + \delta) \cap S$,

$$\begin{aligned}
|f(x) - f(x_0)| &= |f(x) - f_N(x) + f_N(x) - f_N(x_0) + f_N(x_0) - f(x_0)| \\
&\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\
&< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.
\end{aligned}$$

Therefore, f is continuous at x_0. $\square$

The following corollary is immediate from Theorem 9.11.

COROLLARY 9.12. *If f_n is a sequence of continuous functions converging uniformly to f on S, then f is continuous.*

Example 9.1 shows that continuity is not preserved under pointwise convergence. Corollary 9.12 establishes that if S is compact, then $C(S)$ is complete under the uniform metric.

The fact that $C([a, b])$ is closed under uniform convergence is often useful because, given a "bad" function $f \in C([a, b])$, it's often possible to find a sequence f_n of "good" functions in $C([a, b])$ converging uniformly to f. Following is the most widely used theorem of this type.

THEOREM 9.13 (Weierstrass Approximation Theorem). *If $f \in C([a, b])$, then there is a sequence of polynomials $p_n \rightrightarrows f$.*

To prove this theorem, we first need a lemma.

LEMMA 9.14. *For $n \in \mathbb{N}$ let $c_n = \left(\int_{-1}^1 (1-t^2)^n\,dt\right)^{-1}$ and*

$$k_n(t) = \begin{cases} c_n(1-t^2)^n, & |t| \le 1 \\ 0, & |t| > 1 \end{cases}.$$

(See Figure 7.) Then

(a) *$k_n(t) \ge 0$ for all $t \in \mathbb{R}$ and $n \in \mathbb{N}$;*
(b) *$\int_{-1}^1 k_n = 1$ for all $n \in \mathbb{N}$; and,*
(c) *if $0 < \delta < 1$, then $k_n \rightrightarrows 0$ on $(-\infty, -\delta] \cup [\delta, \infty)$.*

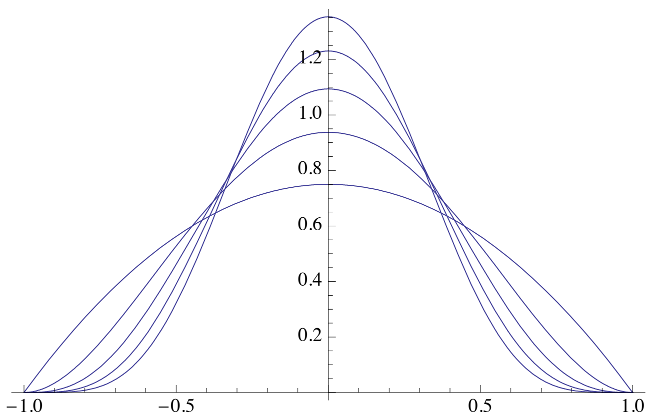

FIGURE 7. Here are the graphs of $k_n(t)$ for $n = 1,2,3,4,5$.

PROOF. Parts (a) and (b) follow easily from the definition of k_n.
To prove (c) first note that

$$1 = \int_{-1}^1 k_n \ge \int_{-1/\sqrt{n}}^{1/\sqrt{n}} c_n(1-t^2)^n\,dt \ge c_n \frac{2}{\sqrt{n}} \left(1 - \frac{1}{n}\right)^n.$$

Since $\left(1 - \frac{1}{n}\right)^n \uparrow \frac{1}{e}$, it follows there is an $\alpha > 0$ such that $c_n < \alpha\sqrt{n}$.[2] Letting $\delta \in (0,1)$ and $\delta \le t \le 1$,

$$k_n(t) \le k_n(\delta) \le \alpha\sqrt{n}(1-\delta^2)^n \to 0$$

by L'Hospital's Rule. Since k_n is an even function, this establishes (c). □

[2] Repeated application of integration by parts shows

$$c_n = \frac{n+1/2}{n} \times \frac{n-1/2}{n-1} \times \frac{n-3/2}{n-2} \times \cdots \times \frac{3/2}{1} = \frac{\Gamma(n+3/2)}{\sqrt{\pi}\,\Gamma(n+1)}.$$

With the aid of Stirling's formula, it can be shown $c_n \approx 0.565\sqrt{n}$.

A sequence of functions satisfying conditions such as those in Lemma 9.14 is called a *convolution kernel* or a *Dirac sequence*.[3] Several such kernels play a key role in the study of Fourier series, as we will see in Theorems 10.5 and 10.13. The one defined above is called the Landau kernel.[4]

We now turn to the proof of the theorem.

PROOF. There is no generality lost in assuming $[a, b] = [0, 1]$, for otherwise we consider the linear change of variables $g(x) = f((b-a)x + a)$. Similarly, we can assume $f(0) = f(1) = 0$, for otherwise we consider $g(x) = f(x) - ((f(1) - f(0))x - f(0)$, which is a polynomial added to f. We can further assume $f(x) = 0$ when $x \notin [0, 1]$.

Set

$$p_n(x) = \int_{-1}^{1} f(x+t)k_n(t)\,dt. \tag{82}$$

To see p_n is a polynomial, change variables in the integral using $u = x + t$ to arrive at

$$p_n(x) = \int_{x-1}^{x+1} f(u)k_n(u-x)\,du = \int_{0}^{1} f(u)k_n(x-u)\,du,$$

because $f(x) = 0$ when $x \notin [0, 1]$. Notice that $k_n(x - u)$ is a polynomial in u with coefficients being polynomials in x, so integrating $f(u)k_n(x - u)$ yields a polynomial in x. (Just try it for a small value of n and a simple function f!)

Use (82) and Lemma 9.14(b) to see for $\delta \in (0, 1)$ that

$$\begin{aligned}
|p_n(x) - f(x)| &= \left|\int_{-1}^{1} f(x+t)k_n(t)\,dt - f(x)\right| \\
&= \left|\int_{-1}^{1} (f(x+t) - f(x))k_n(t)\,dt\right| \\
&\le \int_{-1}^{1} |f(x+t) - f(x)|k_n(t)\,dt \\
&= \int_{-\delta}^{\delta} |f(x+t) - f(x)|k_n(t)\,dt + \int_{\delta<|t|\le 1} |f(x+t) - f(x)|k_n(t)\,dt.
\end{aligned} \tag{83}$$

We'll handle each of the final integrals in turn.

Let $\varepsilon > 0$ and use the uniform continuity of f to choose a $\delta \in (0, 1)$ such that when $|t| < \delta$, then $|f(x+t) - f(x)| < \varepsilon/2$. Then, using Lemma 9.14(b) again,

$$\int_{-\delta}^{\delta} |f(x+t) - f(x)|k_n(t)\,dt < \frac{\varepsilon}{2}\int_{-\delta}^{\delta} k_n(t)\,dt < \frac{\varepsilon}{2} \tag{84}$$

[3] Given two functions f and g defined on $\mathbb{R}$, the *convolution* of f and g is the integral

$$f \star g(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,dt.$$

The term convolution kernel is used because such kernels typically replace g in the convolution given above, as can be seen in the proof of the Weierstrass approximation theorem.

[4] It was investigated by the German mathematician Edmund Landau (1877–1938).

According to Lemma 9.14(c), there is an $N \in \mathbb{N}$ so that when $n \geq N$ and $|t| \geq \delta$, then $k_n(t) < \frac{\varepsilon}{8(\|f\|+1)(1-\delta)}$. Using this, it follows that

$$\begin{aligned}
(85)\quad & \int_{\delta<|t|\leq 1} |f(x+t)-f(x)|k_n(t)\,dt \\
&= \int_{-1}^{-\delta} |f(x+t)-f(x)|k_n(t)\,dt + \int_{\delta}^{1} |f(x+t)-f(x)|k_n(t)\,dt \\
&\leq 2\|f\|\int_{-1}^{-\delta} k_n(t)\,dt + 2\|f\|\int_{\delta}^{1} k_n(t)\,dt \\
&< 2\|f\|\frac{\varepsilon}{8(\|f\|+1)(1-\delta)}(1-\delta) + 2\|f\|\frac{\varepsilon}{8(\|f\|+1)(1-\delta)}(1-\delta) = \frac{\varepsilon}{2}
\end{aligned}$$

Combining (84) and (85), it follows from (83) that $|p_n(x)-f(x)| < \varepsilon$ for all $x \in [0,1]$ and $p_n \rightrightarrows f$. $\square$

COROLLARY 9.15. *If $f \in C([a,b])$ and $\varepsilon > 0$, then there is a polynomial p such that $\|f - p\|_{[a,b]} < \varepsilon$.*

The theorems of this section can also be used to construct some striking examples of functions with unwelcome behavior. Following is perhaps the most famous.

EXAMPLE 9.9. There is a continuous $f : \mathbb{R} \to \mathbb{R}$ that is differentiable nowhere.

PROOF. Thinking of the canonical example of a continuous function that fails to be differentiable at a point—the absolute value function—we start with a "sawtooth" function. (See Figure 8.)

$$s_0(x) = \begin{cases} x - 2n, & 2n \leq x < 2n+1,\ n \in \mathbb{Z} \\ 2n+2-x, & 2n+1 \leq x < 2n+2,\ n \in \mathbb{Z} \end{cases}$$

Notice that s_0 is continuous and periodic with period 2 and maximum value 1. Compress it both vertically and horizontally:

$$s_n(x) = \left(\frac{3}{4}\right)^n s_0\left(4^n x\right),\ n \in \mathbb{N}.$$

Each s_n is continuous and periodic with period $p_n = 2/4^n$ and $\|s_n\| = (3/4)^n$.

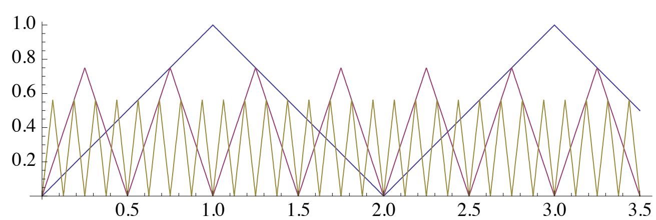

FIGURE 8. s_0, s_1 and s_2 from Example 9.9.

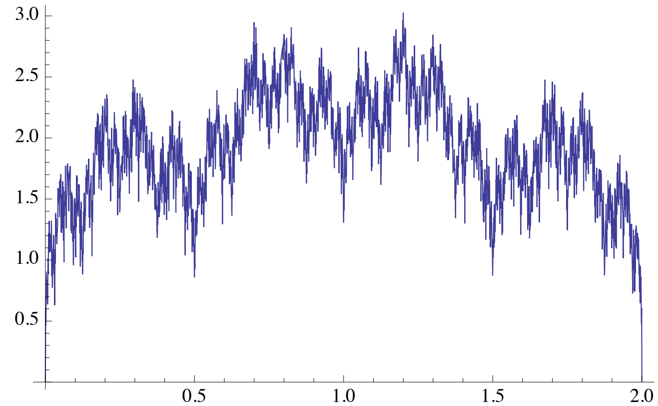

FIGURE 9. The nowhere differentiable function f from Example 9.9.

Finally, the desired function is

$$f(x) = \sum_{n=0}^{\infty} s_n(x).$$

Since $\|s_n\| = (3/4)^n$, the Weierstrass M-test implies the series defining f is uniformly convergent and Corollary 9.12 shows f is continuous on $\mathbb{R}$. We will show f is differentiable nowhere.

Let $x \in \mathbb{R}$, $m \in \mathbb{N}$ and $h_m = 1/(2 \cdot 4^m)$.

If $n > m$, then $h_m/p_n = 4^{n-m-1} \in \mathbb{N}$, so $s_n(x \pm h_m) - s_n(x) = 0$ and

$$\frac{f(x \pm h_m) - f(x)}{\pm h_m} = \sum_{k=0}^{m} \frac{s_k(x \pm h_m) - s_k(x)}{\pm h_m}. \tag{86}$$

On the other hand, if $n < m$, then a worst-case estimate is that

$$\left| \frac{s_n(x \pm h_m) - s_n(x)}{h_m} \right| \le \left(\frac{3}{4}\right)^n / \left(\frac{1}{4^n}\right) = 3^n.$$

This gives

$$\begin{aligned} \left| \sum_{k=0}^{m-1} \frac{s_k(x \pm h_m) - s_k(x)}{\pm h_m} \right| &\le \sum_{k=0}^{m-1} \left| \frac{s_k(x \pm h_m) - s_k(x)}{\pm h_m} \right| \\ &\le \frac{3^m - 1}{3 - 1} < \frac{3^m}{2}. \end{aligned} \tag{87}$$

Since s_m is linear on intervals of length $4^{-m} = 2 \cdot h_m$ with slope $\pm 3^m$ on those linear segments, at least one of the following is true:

$$\left| \frac{s_m(x + h_m) - s(x)}{h_m} \right| = 3^m \text{ or } \left| \frac{s_m(x - h_m) - s(x)}{-h_m} \right| = 3^m. \tag{88}$$

Suppose the first of these is true. The argument is essentially the same in the second case.

Using (86), (87) and (88), the following estimate ensues

$$\begin{aligned}\left|\frac{f(x+h_m)-f(x)}{h_m}\right| &= \left|\sum_{k=0}^{\infty}\frac{s_k(x+h_m)-s_k(x)}{h_m}\right| \\ &= \left|\sum_{k=0}^{m}\frac{s_k(x+h_m)-s_k(x)}{h_m}\right| \\ &\ge \left|\frac{s_m(x+h_m)-s_m(x)}{h_m}\right| - \sum_{k=0}^{m-1}\left|\frac{s_k(x\pm h_m)-s_k(x)}{\pm h_m}\right| \\ &> 3^m - \frac{3^m}{2} = \frac{3^m}{2}.\end{aligned}$$

Since $3^m/2 \to \infty$, it is apparent $f'(x)$ does not exist. □

There are many other constructions of nowhere differentiable continuous functions. The first was published by Weierstrass [20] in 1872, although it was known in the folklore sense among mathematicians earlier than this. (There is an English translation of Weierstrass' paper in [10].) In fact, it is now known in a technical sense that the "typical" continuous function is nowhere differentiable [4].

6. Integration and Uniform Convergence

One of the recurring questions with integrals is when it is true that

$$\lim_{n\to\infty}\int f_n = \int \lim_{n\to\infty} f_n.$$

This is often referred to as "passing the limit through the integral." At some point in her career, any student of advanced analysis or probability theory will be tempted to just blithely pass the limit through. But functions such as those of Example 9.3 show that some care is needed. A common criterion for doing so is uniform convergence.

THEOREM 9.16. *If $f_n : [a,b] \to \mathbb{R}$ such that $\int_a^b f_n$ exists for each n and $f_n \rightrightarrows f$ on $[a,b]$, then*

$$\int_a^b f = \lim_{n\to\infty}\int_a^b f_n$$

.

PROOF. Some care must be taken in this proof, because there are actually two things to prove. Before the equality can be shown, it must be proved that f is integrable.

To show that f is integrable, let $\varepsilon > 0$ and $N \in \mathbb{N}$ such that $\|f - f_N\| < \varepsilon/3(b-a)$. If $P \in \operatorname{part}([a,b])$, then

$$
\begin{aligned}
(89) \qquad |\mathscr{R}(f, P, x_k^*) - \mathscr{R}(f_N, P, x_k^*)| &= |\sum_{k=1}^{n} f(x_k^*)|I_k| - \sum_{k=1}^{n} f_N(x_k^*)|I_k|| \\
&= |\sum_{k=1}^{N} (f(x_k^*) - f_N(x_k^*))|I_k|| \\
&\le \sum_{k=1}^{N} |f(x_k^*) - f_N(x_k^*)||I_k| \\
&< \frac{\varepsilon}{3(b-a))} \sum_{k=1}^{n} |I_k| \\
&= \frac{\varepsilon}{3}
\end{aligned}
$$

According to Theorem 8.10, there is a $P \in \operatorname{part}([a,b])$ such that whenever $P \ll Q_1$ and $P \ll Q_2$, then

$$
(90) \qquad |\mathscr{R}(f_N, Q_1, x_k^*) - \mathscr{R}(f_N, Q_2, y_k^*)| < \frac{\varepsilon}{3}.
$$

Combining (89) and (90) yields

$$
\begin{aligned}
&\left|\mathscr{R}(f, Q_1, x_k^*) - \mathscr{R}(f, Q_2, y_k^*)\right| \\
&\qquad = \left|\mathscr{R}(f, Q_1, x_k^*) - \mathscr{R}(f_N, Q_1, x_k^*) + \mathscr{R}(f_N, Q_1, x_k^*)\right. \\
&\qquad\qquad \left. - \mathscr{R}(f_N, Q_1, x_k^*) + \mathscr{R}(f_N, Q_2, y_k^*) - \mathscr{R}(f, Q_2, y_k^*)\right| \\
&\quad \le \left|\mathscr{R}(f, Q_1, x_k^*) - \mathscr{R}(f_N, Q_1, x_k^*)\right| + \left|\mathscr{R}(f_N, Q_1, x_k^*) - \mathscr{R}(f_N, Q_1, x_k^*)\right| \\
&\qquad\qquad + \left|\mathscr{R}(f_N, Q_2, y_k^*) - \mathscr{R}(f, Q_2, y_k^*)\right| \\
&\qquad\qquad < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\end{aligned}
$$

Another application of Theorem 8.10 shows that f is integrable.

Finally, when $n \ge N$,

$$
\left|\int_a^b f - \int_a^b f_n\right| = \left|\int_a^b (f - f_n)\right| < \int_a^b \frac{\varepsilon}{3(b-a)} = \frac{\varepsilon}{3} < \varepsilon
$$

shows that $\int_a^b f_n \to \int_a^b f$. □

COROLLARY 9.17. *If $\sum_{n=1}^\infty f_n$ is a series of integrable functions converging uniformly on $[a,b]$, then*

$$
\int_a^b \sum_{n=1}^\infty f_n = \sum_{n=1}^\infty \int_a^b f_n
$$

EXAMPLE 9.10. It was shown in Example 4.2 that the geometric series

$$
\sum_{n=0}^\infty t^n = \frac{1}{1-t}, \quad -1 < t < 1.
$$

In Exercise 9.3, you are asked to prove this convergence is uniform on any compact subset of $(-1,1)$. Substituting $-t$ for t in the above formula, it follows that

$$\sum_{n=0}^{\infty}(-t)^n \rightrightarrows \frac{1}{1+t}$$

on $[0,x]$, when $0<x<1$. Corollary 9.17 implies

$$\ln(1+x)=\int_0^x \frac{dt}{1+t}=\sum_{n=0}^{\infty}\int_0^x (-t)^n\,dt = x-x^2+x^3-x^4+\cdots.$$

The same argument works when $-1<x<0$, so

$$\ln(1+x)=x-x^2+x^3-x^4+\cdots$$

when $x\in(-1,1)$.

Combining Theorem 9.16 with Dini's Theorem, gives the following.

COROLLARY 9.18. *If f_n is a sequence of continuous functions converging monotonically to a continuous function f on $[a,b]$, then $\int_a^b f_n \to \int_a^b f$.*

7. Differentiation and Uniform Convergence

The relationship between uniform convergence and differentiation is somewhat more complex than those we've already examined. First, because there are two sequences involved, f_n and f_n', either of which may converge or diverge at a point; and second, because differentiation is more "delicate" than continuity or integration.

Example 9.4 is an explicit example of a sequence of differentiable functions converging uniformly to a function which is not differentiable at a point. The derivatives of the functions from that example converge pointwise to a function that is not a derivative. The Weierstrass Approximation Theorem and Example 9.9 push this to the extreme by showing the existence of a sequence of polynomials converging uniformly to a continuous nowhere differentiable function.

The following theorem starts to shed some light on the situation.

THEOREM 9.19. *If f_n is a sequence of derivatives defined on $[a,b]$ and $f_n \rightrightarrows f$, then f is a derivative.*

PROOF. For each n, let F_n be an antiderivative of f_n. By considering $F_n(x)-F_n(a)$, if necessary, there is no generality lost with the assumption that $F_n(a)=0$.

Let $\varepsilon>0$. There is an $N\in\mathbb{N}$ such that

$$m,n\ge N \implies \|f_m-f_n\|<\frac{\varepsilon}{b-a}.$$

If $x\in[a,b]$ and $m,n\ge N$, then the Mean Value Theorem and the assumption that $F_m(a)=F_n(a)=0$ yield a $c\in[a,b]$ such that

$$\begin{aligned}|F_m(x)-F_n(x)| &= |(F_m(x)-F_n(x))-(F_m(a)-F_n(a))|\\ &= |f_m(c)-f_n(c)|\,|x-a| \le \|f_m-f_n\|(b-a)<\varepsilon. \qquad (91)\end{aligned}$$

This shows F_n is a Cauchy sequence in $C([a,b])$ and there is an $F\in C([a,b])$ with $F_n \rightrightarrows F$.

It suffices to show $F' = f$. To do this, several estimates are established. Let $M \in \mathbb{N}$ so that

$$m, n \geq M \implies \|f_m - f_n\| < \frac{\varepsilon}{3}.$$

Notice this implies

$$\|f - f_n\| \leq \frac{\varepsilon}{3}, \ \forall n \geq M. \tag{92}$$

For such $m, n \geq M$ and $x, y \in [a, b]$ with $x \neq y$, another application of the Mean Value Theorem gives

$$\begin{aligned}
&\left| \frac{F_n(x) - F_n(y)}{x - y} - \frac{F_m(x) - F_m(y)}{x - y} \right| \\
&\qquad = \frac{1}{|x - y|} \left| (F_n(x) - F_m(x)) - (F_n(y) - F_m(y)) \right| \\
&\qquad\qquad = \frac{1}{|x - y|} \left| f_n(c) - f_m(c) \right| |x - y| \leq \|f_n - f_m\| < \frac{\varepsilon}{3}.
\end{aligned}$$

Letting $m \to \infty$, it follows that

$$\left| \frac{F_n(x) - F_n(y)}{x - y} - \frac{F(x) - F(y)}{x - y} \right| \leq \frac{\varepsilon}{3}, \ \forall n \geq M. \tag{93}$$

Fix $n \geq M$ and $x \in [a, b]$. Since $F_n'(x) = f_n(x)$, there is a $\delta > 0$ so that

$$\left| \frac{F_n(x) - F_n(y)}{x - y} - f_n(x) \right| < \frac{\varepsilon}{3}, \forall y \in (x - \delta, x + \delta) \setminus \{x\}. \tag{94}$$

Finally, using (93), (94) and (92), we see

$$\begin{aligned}
&\left| \frac{F(x) - F(y)}{x - y} - f(x) \right| \\
&\quad = \left| \frac{F(x) - F(y)}{x - y} - \frac{F_n(x) - F_n(y)}{x - y} \right. \\
&\qquad\qquad \left. + \frac{F_n(x) - F_n(y)}{x - y} - f_n(x) + f_n(x) - f(x) \right| \\
&\quad \leq \left| \frac{F(x) - F(y)}{x - y} - \frac{F_n(x) - F_n(y)}{x - y} \right| \\
&\qquad\qquad + \left| \frac{F_n(x) - F_n(y)}{x - y} - f_n(x) \right| + \left| f_n(x) - f(x) \right| \\
&\qquad\qquad\qquad < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

This establishes that

$$\lim_{y \to x} \frac{F(x) - F(y)}{x - y} = f(x),$$

as desired. □

COROLLARY 9.20. *If $G_n \in C([a, b])$ is a sequence such that $G_n' \rightrightarrows g$ and $G_n(x_0)$ converges for some $x_0 \in [a, b]$, then $G_n \rightrightarrows G$ where $G' = g$.*

PROOF. Suppose $G_n(x_0) \to \alpha$. For each n choose an antiderivative F_n of g_n such that $F_n(a) = 0$. Theorem 9.19 shows g is a derivative and an argument similar to that in the proof of Theorem 9.19 shows $F_n \rightrightarrows F$ on $[a, b]$, where $F' = g$. Since $F'_n - G'_n = 0$, Corollary (7.16) shows $G_n(x) = F_n(x) + (G_n(x_0) - F_n(x_0))$. Define $G(x) = F(x) + (\alpha - F(x_0))$.

Let $\varepsilon > 0$ and $x \in [a, b]$. There is an $N \in \mathbb{N}$ such that

$$n \ge N \implies \|F_n - F\| < \frac{\varepsilon}{3} \text{ and } |G_n(x_0) - \alpha| < \frac{\varepsilon}{3}.$$

If $n \ge N$,

$$\begin{aligned}
|G_n(x) - G(x)| &= |F_n(x) + (G_n(x_0) - F_n(x_0)) - (F(x) + (\alpha - F(x_0)))| \\
&\le |F_n(x) - F(x)| + |G_n(x_0) - \alpha| + |F_n(x_0) - F(x_0)| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon
\end{aligned}$$

This shows $G_n \rightrightarrows G$ on $[a, b]$ where $G' = F' = g$. □

COROLLARY 9.21. *If f_n is a sequence of differentiable functions defined on $[a, b]$ such that $\sum_{k=1}^{\infty} f_k(x_0)$ exists for some $x_0 \in [a, b]$ and $\sum_{k=1}^{\infty} f'_k$ converges uniformly, then*

$$\left(\sum_{k=1}^{\infty} f\right)' = \sum_{k=1}^{\infty} f'$$

PROOF. Left as an exercise. □

EXAMPLE 9.11. Let $a > 0$ and $f_n(x) = x^n/n!$. Note that $f'_n = f_{n-1}$ for $n \in \mathbb{N}$. Example 9.8 shows $\sum_{n=0}^{\infty} f'_n(x)$ is uniformly convergent on $[-a, a]$. Corollary 9.21 shows

$$\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)' = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{95}$$

on $[-a, a]$. Since a is an arbitrary positive constant, (95) is seen to hold on all of $\mathbb{R}$.

If $f(x) = \sum_{n=0}^{\infty} x^n/n!$, then the argument given above implies the initial value problem

$$\begin{cases} f'(x) = f(x) \\ f(0) = 1 \end{cases}$$

As is well-known, the unique solution to this problem is $f(x) = e^x$. Therefore,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

8. Power Series

8.1. The Radius and Interval of Convergence. One place where uniform convergence plays a key role is with power series. Recall the definition.

DEFINITION 9.22. A *power series* is a function of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n. \tag{96}$$

Members of the sequence a_n are the *coefficients* of the series. The domain of f is the set of all x at which the series converges. The constant c is called the *center* of the series.

To determine the domain of (96), let $x \in \mathbb{R} \setminus \{c\}$ and use the root test to see the series converges when

$$\limsup |a_n (x-c)^n|^{1/n} = |x-c| \limsup |a_n|^{1/n} < 1$$

and diverges when

$$|x-c| \limsup |a_n|^{1/n} > 1.$$

If $r \limsup |a_n|^{1/n} \le 1$ for some $r \ge 0$, then these inequalities imply (96) is absolutely convergent when $|x-c| < r$. In other words, if

$$R = \text{lub}\,\{r : r \limsup |a_n|^{1/n} < 1\}, \tag{97}$$

then the domain of (96) is an interval of radius R centered at c. The root test gives no information about convergence when $|x-c| = R$. This R is called the *radius of convergence* of the power series. Assuming $R > 0$, the open interval centered at c with radius R is called the *interval of convergence*. It may be different from the domain of the series because the series may converge at one endpoint or both endpoints of the interval of convergence.

The ratio test can also be used to determine the radius of convergence, but, as shown in (31), it will not work as often as the root test. When it does,

$$R = \text{lub}\,\{r : r \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1\}. \tag{98}$$

This is usually easier to compute than (97), and both will give the same value for R.

EXAMPLE 9.12. Calling to mind Example 4.2, it is apparent the geometric power series $\sum_{n=0}^{\infty} x^n$ has center 0, radius of convergence 1 and domain $(-1,1)$.

EXAMPLE 9.13. For the power series $\sum_{n=1}^{\infty} 2^n (x+2)^n / n$, we compute

$$\limsup \left(\frac{2^n}{n} \right)^{1/n} = 2 \implies R = \frac{1}{2}.$$

Since the series diverges when $x = -2 \pm \frac{1}{2}$, it follows that the interval of convergence is $(-5/2, -3/2)$.

EXAMPLE 9.14. The power series $\sum_{n=1}^{\infty} x^n / n$ has interval of convergence $(-1,1)$ and domain $[-1,1)$. Notice it is not absolutely convergent when $x = -1$.

EXAMPLE 9.15. The power series $\sum_{n=1}^{\infty} x^n / n^2$ has interval of convergence $(-1,1)$, domain $[-1,1]$ and is absolutely convergent on its whole domain.

The preceding is summarized in the following theorem.

THEOREM 9.23. *Let the power series be as in (96) and R be given by either (97) or (98).*

(a) *If $R = 0$, then the domain of the series is $\{c\}$.*
(b) *If $R > 0$ the series converges absolutely at x when $|c - x| < R$ and diverges at x when $|c-x| > R$. In the case when $R = \infty$, the series converges everywhere.*
(c) *If $R \in (0,\infty)$, then the series may converge at none, one or both of $c - R$ and $c + R$.*

8.2. Uniform Convergence of Power Series. The partial sums of a power series are a sequence of polynomials converging pointwise on the domain of the series. As has been seen, pointwise convergence is not enough to say much about the behavior of the power series. The following theorem opens the door to a lot more.

THEOREM 9.24. *A power series converges absolutely and uniformly on compact subsets of its interval of convergence.*

PROOF. There is no generality lost in assuming the series has the form of (96) with $c = 0$. Let the radius of convergence be $R > 0$ and K be a compact subset of $(-R, R)$ with $\alpha = \operatorname{lub}\{|x| : x \in K\}$. Choose $r \in (\alpha, R)$. If $x \in K$, then $|a_n x^n| < |a_n r^n|$ for $n \in \mathbb{N}$. Since $\sum_{n=0}^{\infty} |a_n r^n|$ converges, the Weierstrass M-test shows $\sum_{n=0}^{\infty} a_n x^n$ is absolutely and uniformly convergent on K. □

The following two corollaries are immediate consequences of Corollary 9.12 and Theorem 9.16, respectively.

COROLLARY 9.25. *A power series is continuous on its interval of convergence.*

COROLLARY 9.26. *If $[a, b]$ is an interval contained in the interval of convergence for the power series $\sum_{n=0}^{\infty} a_n (x - c)^n$, then*

$$\int_a^b \sum_{n=0}^{\infty} a_n (x-c)^n = \sum_{n=0}^{\infty} a_n \int_a^b (x-c)^n.$$

EXAMPLE 9.16. Define

$$f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}.$$

Since $\lim_{x\to 0} f(x) = 1$, f is continuous everywhere. Suppose we want $\int_0^{\pi} f$ with an accuracy of five decimal places.

If $x \neq 0$,

$$f(x) = \frac{1}{x} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)!} x^{2n-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n}$$

The latter series converges to f everywhere. Corollary 9.26 implies

$$
\begin{aligned}
\int_0^\pi f(x)\,dx &= \int_0^\pi \left(\sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!}x^{2n}\right)dx \\
&= \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!}\int_0^\pi x^{2n}\,dx \\
&= \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)(2n+1)!}\pi^{2n+1} \qquad (99)
\end{aligned}
$$

The latter series satisfies the Alternating Series Test. Since $\pi^15/(15\times 15!)\approx 1.5\times 10^{-6}$, Corollary 4.20 shows

$$\int_0^\pi f(x)\,dx \approx \sum_{n=0}^6 \frac{(-1)^n}{(2n+1)(2n+1)!}\pi^{2n+1} \approx 1.85194$$

The next question is: What about differentiability?

Notice that the continuity of the exponential function and L'Hospital's Rule give

$$\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty}\exp\left(\frac{\ln n}{n}\right) = \exp\left(\lim_{n\to\infty}\frac{\ln n}{n}\right) = \exp(0) = 1.$$

Therefore, for any sequence a_n,

$$\limsup (na_n)^{1/n} = \limsup n^{1/n}a_n^{1/n} = \limsup a_n^{1/n}. \qquad (100)$$

Now, suppose the power series $\sum_{n=0}^\infty a_n x^n$ has a nontrivial interval of convergence, I. Formally differentiating the power series term-by-term gives a new power series $\sum_{n=1}^\infty n a_n x^{n-1}$. According to (100) and Theorem 9.23, the term-by-term differentiated series has the same interval of convergence as the original. Its partial sums are the derivatives of the partial sums of the original series and Theorem 9.24 guarantees they converge uniformly on any compact subset of I. Corollary 9.21 shows

$$\frac{d}{dx}\sum_{n=0}^\infty a_n x^n = \sum_{n=0}^\infty \frac{d}{dx}a_n x^n = \sum_{n=1}^\infty n a_n x^{n-1},\ \forall x\in I.$$

This process can be continued inductively to obtain the same results for all higher order derivatives. We have proved the following theorem.

THEOREM 9.27. *If $f(x)=\sum_{n=0}^\infty a_n(x-c)^n$ is a power series with nontrivial interval of convergence, I, then f is differentiable to all orders on I with*

$$f^{(m)}(x) = \sum_{n=m}^\infty \frac{n!}{(n-m)!}a_n(x-c)^{n-m}. \qquad (101)$$

Moreover, the differentiated series has I as its interval of convergence.

8.3. Taylor Series. Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has $I = (-R, R)$ as its interval of convergence for some $R > 0$. According to Theorem 9.27,

$$f^{(m)}(0) = \frac{m!}{(m-m)!} a_m \implies a_m = \frac{f^{(m)}(0)}{m!}, \ \forall m \in \omega.$$

Therefore,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n, \ \forall x \in I.$$

This is a remarkable result! It shows that the values of f on I are completely determined by its values on any neighborhood of 0. This is summarized in the following theorem.

THEOREM 9.28. *If a power series $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ has nontrivial interval of convergence I, then*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n, \ \forall x \in I. \tag{102}$$

The series (102) is called the *Taylor series*[5] for f centered at c. The Taylor series can be formally defined for any function that has derivatives of all orders at c, but, as Example 7.9 shows, there is no guarantee it will converge to the function anywhere except at c. Taylor's Theorem 7.18 can be used to examine the question of pointwise convergence. If f can be represented by a power series on an open interval I, then f is said to be *analytic* on I.

8.4. The Endpoints of the Interval of Convergence. We have seen that at the endpoints of its interval of convergence a power series may diverge or even absolutely converge. A natural question when it does converge is the following: What is the relationship between the value at the endpoint and the values inside the interval of convergence?

THEOREM 9.29 (Abel). *A power series is continuous on its domain.*

PROOF. Let $f(x) = \sum_{n=0}^{\infty} (x-c)^n$ have radius of convergence R and interval of convergence I. If $I = \{0\}$, the theorem is vacuously true from Definition 6.9. If $I = \mathbb{R}$, the theorem follows from Corollary 9.25. So, assume $R \in (0, \infty)$. It must be shown that if f converges at an endpoint of $I = (c-R, c+R)$, then f is continuous at that endpoint.

It can be assumed $c = 0$ and $R = 1$. There is no loss of generality with either of these assumptions because otherwise just replace $f(x)$ with $f((x+c)/R)$. The theorem will be proved for $\alpha = c + R$ since the other case is proved similarly.

[5] When $c = 0$, it is often called the *Maclaurin series* for f.

Set $s = f(1)$, $s_{-1} = 0$ and $s_n = \sum_{k=0}^{n} a_k$ for $n \in \omega$. For $|x| < 1$,

$$\begin{aligned}\sum_{k=0}^{n} a_k x^k &= \sum_{k=0}^{n} (s_k - s_{k-1}) x^k \\ &= \sum_{k=0}^{n} s_k x^k - \sum_{k=1}^{n} s_{k-1} x^k \\ &= s_n x^n + \sum_{k=0}^{n-1} s_k x^k - x \sum_{k=0}^{n-1} s_k x^k \\ &= s_n x^n + (1-x) \sum_{k=0}^{n-1} s_k x^k\end{aligned}$$

When $n \to \infty$, since s_n is bounded and $|x| < 1$,

$$f(x) = (1-x) \sum_{k=0}^{\infty} s_k x^k. \tag{103}$$

Since $(1-x)\sum_{n=0}^{\infty} x^n = 1$, (103) implies

$$|f(x) - s| = \left| (1-x) \sum_{k=0}^{\infty} (s_k - s) x^k \right|. \tag{104}$$

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that whenever $n \geq N$, then $|s_n - s| < \varepsilon/2$. Choose $\delta \in (0,1)$ so

$$\delta \sum_{k=0}^{N} |s_k - s| < \varepsilon/2.$$

Suppose x is such that $1 - \delta < x < 1$. With these choices, (104) becomes

$$\begin{aligned}|f(x) - s| &\leq \left| (1-x) \sum_{k=0}^{N} (s_k - s) x^k \right| + \left| (1-x) \sum_{k=N+1}^{\infty} (s_k - s) x^k \right| \\ &< \delta \sum_{k=0}^{N} |s_k - s| + \frac{\varepsilon}{2} \left| (1-x) \sum_{k=N+1}^{\infty} x^k \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

It has been shown that $\lim_{x \uparrow 1} f(x) = f(1)$, so $1 \in C(f)$. □

Here is an example showing the power of these techniques.

EXAMPLE 9.17. The series

$$\sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2}$$

has $(-1,1)$ as its interval of convergence. If $0 \leq |x| < 1$, then Corollary 9.17 justifies

$$\arctan(x) = \int_0^x \frac{dt}{1+t^2} = \int_0^x \sum_{n=0}^{\infty} (-1)^n t^{2n}\, dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}.$$

This series for the arctangent converges by the alternating series test when $x = 1$, so Theorem 9.29 implies

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \lim_{x \uparrow 1} \arctan(x) = \arctan(1) = \frac{\pi}{4}. \tag{105}$$

A bit of rearranging gives the formula

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right),$$

which is known as Gregory's series for π.

Finally, Abel's theorem opens up an interesting idea for the summation of series. Suppose $\sum_{n=0}^{\infty} a_n$ is a series. The *Abel sum* of this series is

$$A\sum_{n=0}^{\infty} a_n = \lim_{x \uparrow 1} \sum_{n=0}^{\infty} a_n x^n.$$

Consider the following example.

EXAMPLE 9.18. Let $a_n = (-1)^n$ so

$$\sum_{n=0}^{\infty} a_n = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

diverges. But,

$$A\sum_{n=0}^{\infty} a_n = \lim_{x \uparrow 1} \sum_{n=0}^{\infty} (-x)^n = \lim_{x \uparrow 1} \frac{1}{1+x} = \frac{1}{2}.$$

This shows the Abel sum of a series may exist when the ordinary sum does not. Abel's theorem guarantees when both exist they are the same.

Abel summation is one of many different summation methods used in areas such as harmonic analysis. (For another see Exercise 4.4.25.)

THEOREM 9.30 (Tauber). *If $\sum_{n=0}^{\infty} a_n$ is a series satisfying*

(a) $na_n \to 0$ *and*
(b) $A\sum_{n=0}^{\infty} a_n = A$,

then $\sum_{n=0}^{\infty} a_n = A$.

PROOF. Let $s_n = \sum_{k=0}^{n} a_k$. For $x \in (0,1)$ and $n \in \mathbb{N}$,

$$\begin{aligned}
\left| s_n - \sum_{k=0}^{\infty} a_k x^k \right| &= \left| \sum_{k=0}^{n} a_k - \sum_{k=0}^{n} a_k x^k - \sum_{k=n+1}^{\infty} a_k x^k \right| \\
&= \left| \sum_{k=0}^{n} a_k (1 - x^k) - \sum_{k=n+1}^{\infty} a_k x^k \right| \\
&= \left| \sum_{k=0}^{n} a_k (1-x)(1 + x + \cdots + x^{k-1}) - \sum_{k=n+1}^{\infty} a_k x^k \right| \\
&\le (1-x) \sum_{k=0}^{n} k|a_k| + \sum_{k=n+1}^{\infty} |a_k| x^k. \qquad (106)
\end{aligned}$$

Let $\varepsilon > 0$. According to (a) and Exercise 3.3.21, there is an $N \in \mathbb{N}$ such that

$$n \geq N \implies n|a_n| < \frac{\varepsilon}{2} \text{ and } \frac{1}{n}\sum_{k=0}^{n} k|a_k| < \frac{\varepsilon}{2}. \tag{107}$$

Let $n \geq N$ and $1 - 1/n < x < 1$. Using the right term in (107),

$$(1-x)\sum_{k=0}^{n} k|a_k| < \frac{1}{n}\sum_{k=0}^{n} k|a_k| < \frac{\varepsilon}{2}. \tag{108}$$

Using the left term in (107) gives

$$\begin{aligned} \sum_{k=n+1}^{\infty} |a_k|x^k &< \sum_{k=n+1}^{\infty} \frac{\varepsilon}{2k}x^k \\ &< \frac{\varepsilon}{2n}\frac{x^{n+1}}{1-x} \\ &< \frac{\varepsilon}{2}. \end{aligned} \tag{109}$$

Combining (107) and (107) with (106) shows

$$\left| s_n - \sum_{k=0}^{\infty} a_k x^k \right| < \varepsilon.$$

Assumption (b) implies $s_n \to A$. □

9. Exercises

9.1. If $f_n(x) = nx(1-x)^n$ for $0 \leq x \leq 1$, then show f_n converges pointwise, but not uniformly on $[0,1]$.

9.2. Show $\sin^n x$ converges uniformly on $[0,a]$ for all $a \in (0,\pi/2)$. Does $\sin^n x$ converge uniformly on $[0,\pi/2)$?

9.3. Show that $\sum x^n$ converges uniformly on $[-r,r]$ when $0 < r < 1$, but not on $(-1,1)$.

9.4. Prove $\sum_{n=0}^{\infty} x^n/n!$ does not converge uniformly on $\mathbb{R}$.

9.5. The series

$$\sum_{n=0}^{\infty} \frac{\cos nx}{e^{nx}}$$

is uniformly convergent on any set of the form $[a,\infty)$ with $a > 0$.

9.6. A sequence of functions $f_n : S \to \mathbb{R}$ is *uniformly bounded* on S if there is an $M > 0$ such that $\|f_n\|_S \leq M$ for all $n \in \mathbb{N}$. Prove that if f_n is uniformly convergent on S and each f_n is bounded on S, then the sequence f_n is uniformly bounded on S.

9.7. Let $S \subset \mathbb{R}$ and $c \in \mathbb{R}$. If $f_n : S \to \mathbb{R}$ is a Cauchy sequence, then so is cf_n.

9.8. If $S \subset \mathbb{R}$ and $f_n, g_n : S \to \mathbb{R}$ are Cauchy sequences, then so is $f_n + g_n$.

9.9. Let $S \subset \mathbb{R}$. If $f_n, g_n : S \to \mathbb{R}$ are uniformly bounded Cauchy sequences, then so is $f_n g_n$.

9.10. Prove or give a counterexample: If f_n is a sequence of monotone functions converging pointwise to a continuous function f, then $f_n \rightrightarrows f$.

9.11. Prove or give a counterexample: If $f_n : [a,b] \to \mathbb{R}$ is a sequence of monotone functions converging pointwise to a continuous function f, then $f_n \rightrightarrows f$.

9.12. Prove there is a sequence of polynomials on $[a,b]$ converging uniformly to a nowhere differentiable function.

9.13. Prove Corollary 9.21.

9.14. Prove $\pi = 2\sqrt{3}\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(2n+1)}$. (This is the Madhava-Leibniz series which was used in the fourteenth century to compute π to 11 decimal places.)

9.15. If $\exp(x) = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$, then $\frac{d}{dx}\exp(x) = \exp(x)$ for all $x \in \mathbb{R}$.

9.16. Is $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ Abel convergent?